Penultimate Polyhedra

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Introduction

These are some notes that I originally hacked up for my sister. They describe how to make polyhedra out of the "penultimate" module. This module is originally described in Jay Ansill's book "Lifestyle Origami," [Ans92] and he attributes the module to Robert Neale. I have omitted how to put the modules together – buy the book, or figure it out for yourself. It's pretty obvious. The pentagon module is pretty much lifted straight from the book (although I've found 3x4 paper easier to work with than 4x4 paper), but the others are my own tweaks. Neale's designs are nicer because they don't require any cuts. You don't need to cut these either — you can fold away the excess paper — however, if you are using small paper, the resulting polyhedra are far more stable if you do cut them. I am not a purist.

This method of making modules lends itself to many variations besides the ones shown here. All you need is a calculater with trigonometric functions and you can figure them out for yourself. Besides the Platonic and Archimedian solids, I have made various others: rhombic dodecahedron, rhombic triacontahedron, numerous prisms and antiprisms, stella octangula, great and lesser stellated dodecahdra, compound of 5 tetrahedra, compound of 5 octahedra, dual of the snub cube, etc. If you're interested, I can give descriptions of the modules, although perhaps not quickly. I also have pictures of many finished polyhedra online (in gif files) — send me email if you'd like me to send them to you.

The polyhedron numbers referenced below are from the pictures of the Archimedean solids in Fuse's book "Unit Origami" [Fus90]. Kasahara/Takahama's "Origami for the Connoisseur" [KT87] also has pictures of these polyhedra with a different numbering.

I haven't included modules for octagons or decagons. I've made octagonal ones, but they're pretty flimsy, meaning that the resulting polyhedra cannot exist in the same house as cats without the aid of glue or a gun. Of course, that doesn't bother me much. If you can't figure out how to make octagonal or decagonal modules, send me email, and I'll make the diagrams.

Notes

These are some of the polyhedra that you can make with the basic modules (triangle, square, pentagon, hexagon). The ones with octagons and decagons can be made with similar modules, but they're pretty flimsy, so I don't include them. Each module is an edge of the polyhedron. The notation is as follows — if it says "sq–tr", then it means to fold a module with a 90-degree angle on one side, and a 60-degree angle on the other. That edge will be used for places on the polyhedron where a triangle meets a square. For example, on the cuboctahedron, all edges are like this.

Coloring is a matter of taste. I have made most of these polyhedra, and some colorings to look much better than others (at least to me). In general, I've found that it's best to make sure that all three edges of any triangle are not the same color. Two are fine. Three tend to blur the fact that it's a triangle.

- **The Tetrahedron (#1)**. 4 triangular faces. 6 modules. All are tr–tr. The last one is usually difficult to get in. I usually color with 2 edges each of three different colors such that each triangular face is composed of edges of three different colors.
- **The Cube (#2)**. 6 square faces. 12 modules. All are sq–sq. Most any coloration works.
- **The Octahedron (#3)**. 8 triangular faces. 12 modules. All are tr–tr. Most any coloration works.
- **The Dodecahedron (#4)**. 12 pentagonal faces. 30 modules. All are pe–pe. This is a great piece of origami – simple to make, and rock solid. It is a good one upon which to learn how to use these modules There are a couple of neat colorations here. They mostly use ten modules each of three different colors. One way is color such that no two adjacent edges on a pentagonal face are the same. The one I like better is to make the top and bottom pentagons out of color 1. Then have the 10 edges emanating from the top and bottom pentagons be color 2. The remaining ten edges of color 3 form a band around the middle. You can use this same design with two colors by making the band around the middle from color 1. Finally, you can color again with ten modules each of three different colors in the following way: Take five modules of one color, and fit them together as follows:

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  Do that with the remaining pieces so that you have 6 composites like the one above, two of each color. These will fit together to make a dodecahedron with each pair of composites on opposite ends of the dodecahedron.
- **The Icosahedron (#5)**. 20 triangular faces. 30 modules. All are tr–tr. You can color this like the dodecahedron, with ten modules each of three difference colors. It can be colored so that all triangles have edges of each color. Or you can color in a way analgous to the dodecahedron: All triangles meet in groups of five. Take five edges of color 1, and make one vertex of the icosahedron. This will make five incomplete triangles. Complete the triangles with color 2. Repeat this with the remaining five modules of color one, and the remaining five modules of color 2. Now you have made two pentagonal pyramids, which compose the top and the bottom of the icosahedron. Use color 3 for the remaining edges, which make a zig-zag around the middle.
- **Truncated Tetrahedron (#6)**. 4 triangular faces, 4 hexagonal faces. 18 modules: 12 tr–he and 6 he–he. You can color this with three colors as follows: Arrange the he–he modules like they are the

edges of a tetrahedron. Then add the tr–he modules so that all triangles have edges of each color. You can do this so that each hexagon has no adjacent edges of the same color, or so that hexagon edges all come in pairs of the same color.

- **Truncated Cube (#7)**. 6 octagonal faces, 8 triangular faces. 36 modules: 12 oc–oc, 24 tr–oc. I haven't included modules for octagons.

- **Truncated Octahedron (#8)**. 6 square faces, 8 hexagonal faces. 36 modules: 12 he–he, 24 sq–he. This works nicely with two colors – all the sq–he modules are one, and all the he–he are another. Or you can use three colors, evenly divided so that opposite squares are the same color (and all edges in a square are the same color), and modules connecting two squares are of the third color.

- **Truncated Dodecahedron (#9)**. 12 decagonal faces, 20 triangular faces. 90 modules: 30 de–de, 60 tr–de. I have not made this one. It requires decagonal faces.

- **Truncated Icosahedron (#10)**. 12 pentagonal faces, 20 hexagonal faces. 90 modules: 60 pe–he, 30 he–he. This makes a beautiful piece of origami. All the ones I've made have been two colors: one for the pe–he modules, and one for the he–he modules. It is surprisingly sturdy.

- **Cuboctahedron (#11)**. 6 square faces, 8 triangular faces. 24 modules, all tr–sq. There are many nice ways to color this one, for example eight modules of each color forming opposite pair of squares. One neat one is to use six modules each of four colors, having each color form a hexagonal band around the middle of the polyhedron.

- **Icosidodecahedron (#12)**. 12 pentagonal faces, 20 triangular faces. 60 modules, all tr–pe. The best coloration I found for this one was to use 20 modules each of three colors. Take color 1 and make the top and bottom pentagons. Take color 2, and complete the triangles around each of these pentagons. This will take all 20 modules of color 2. Take color 3, and form the two edges of the remaining triangles that attach to the triangles of color 2. This will take all 20 modules of color 3. The remaining 10 modules of color 1 form a decagonal band around the middle of the polyhedron, attaching the two halves you have just created.

  It is also possible to divide the edges of this polyhedron into six decagonal bands. Unfortunately, it is hard to get six colors to look nice together.

- **Rhombicuboctahedron (#13)**. 18 squares and 8 triangles. 48 modules: 24 sq–sq and 24 tr–sq. This is another very pretty solid. I have always used 16 modules each of three colors (8 sq–sq and 8 tr–sq), and had each color form two parallel octagons around the middle.

- **Rhombitruncatedcuboctahedron (#14)**. 12 squares, 8 hexagons, 6 octagons. 72 modules: 24 oc–he, 24 oc–sq, 24 sq–he. I have never made this one.

- **Rhombicosidodecahedron (#15)**. 30 squares, 12 pentagons, 20 triangles. 120 modules: 60 sq–pe, 60 tr–sq. I made this one once with two colors, one for each type of module. It was ugly, because all the triangles were the same color. Next time I make this, I'll probably do one of the following:

  1. Split the modules into three colors (20 sq–pe and 20 tr–sq of each color). Arrange the modules so that each square has edges of the same color. Then arrange the squares so that all trianges have edges of all three colors. This can be done by arranging the squares as if they were edges of a dodecahedron, where each edge of the pentagon is never adjacent to an edge of the same color (this is the first coloration of the dodecahedron suggested above).

2. Use four colors. Color all of the sq–pe modules with color 1. Then divide the remaining modules into 20 each of colors 2, 3, and 4. Now, all the pentagons will be the same color. Make all the triangles have an edge of each color. If you really want to be studly, you can arrange it so that each square has opposite edges of the same color.

3. Do a coloring that is symmetrical from top to bottom. I.e. start by making the top and bottom pentagons color 1. Then make all the edges emanating from them have color 2. Complete the triangles with color 3 (perhaps have color 3 form the decagonal band concentric to the top pentagon). Continue in some similar fasion...

- **Rhombitruncatedicosidodecahedron (#16)**. 30 squares, 12 decagons, 20 hexagons. 180 modules: 60 de–sq, 60 de–he, 60 he–sq. I haven't made this one.

- **Snub Cube (#17)**. 6 squares, 32 triangles. 60 modules: 24 tr–sq and 36 tr–tr. I've made two of these, one with 4 colors and one with three. In the one with three, I colored as follows: Divide both sets of modules into three equal number of colors. With the tr–sq modules, make six squares, two of each color. Take the square of color 1. You'll note from the picture that each vertex of the square has three tr–tr modules incident to it. On one vertex, make these of color 2-1-2. On the next, make them 3-1-3. On the next, make them 2-1-2 again, and on the final vertex, make them 3-1-3 again. This is how it will work with all squares – if a square is of color y, then one pair of opposite vertices will have modules ordered x-y-x, and the other pair will have modules ordered z-y-z. It works out so that each pair of squares is on opposite faces, and the pattern is pleasing.

  To get a snub cube from four colors, simply do the same as above, only make all the tr–sq modules out of color 4. The ordering of the tr–tr modules should be the same.

- **Snub Dodecahedron (#18)**. 12 pentagons and 80 triangles. 150 modules: 60 tr–pe and 90 tr–tr. I haven't made this one yet, but it has been started. I am using 4 colors, dividing both sets of modules equally. You can color the faces of a dodecahedron with three faces each of four colors so that no adjacent faces have the same color. Similarly, I will use the tr–pe modules to make three pentagons of each of the four colors, and align them in the snub dodecahedron so that no adjacent pentagons have the same color. Then I will add the tr–tr modules as follows. You'll note that there are two types of triangles in the snub dodecahedron – those that share one edge with a pentagon, and those that share no edges with pentagons. Consider the second type. Each of the three vertices is shared by a pentagon of a different color. The edges of the triangle will contain those three colors such that the two triangular edges incident to a vertex are different from the pentagon shared at that vertex. This specifies a unique coloring of these triangles (given the coloring of the pentagons). I can't yet see a pattern for the remaining edges, but I imagine it won't be difficult – just have to make it and see.

References

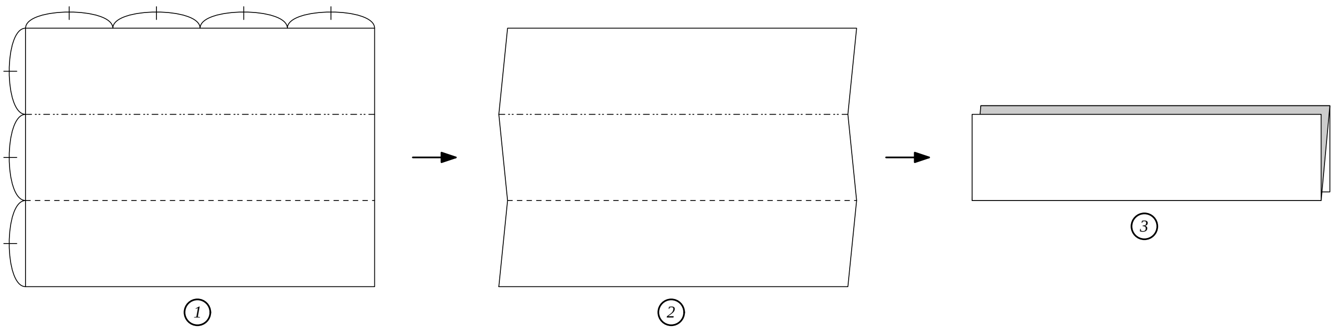
[Ans92] J. Ansill. *Lifestyle Origami.* HarperCollins Publishers, 10 East 53 Street, New York, NY 10022, 1992.

[Fus90] T. Fusè. *Unit Origami.* Japan Publications Inc., Tokyo and New York, 1990.

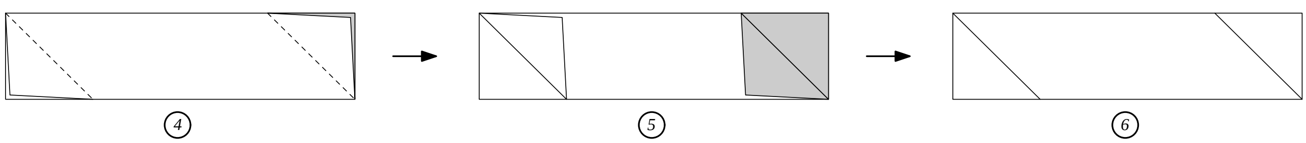
[KT87] K. Kasahara and T. Takahama. *Origami for the Connoisseur.* Japan Publications Inc., Tokyo and New York, 1987.

Pentagon Module (108 Degrees)

Start with a 4x3 rectangle, and collapse like an accordian:

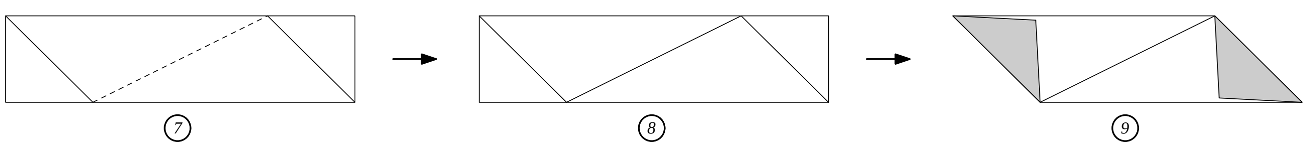


Fold opposite corners in -- use only the top layer -- and unfold

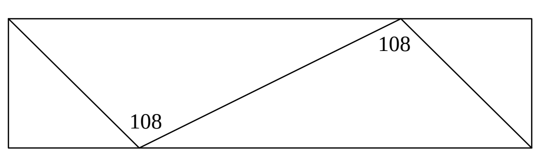


Fold along the dotted line and unfold

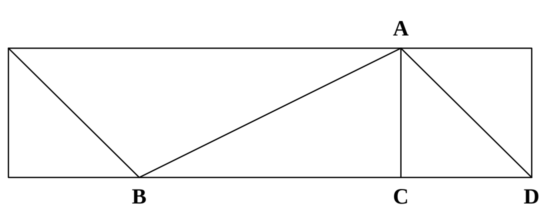
Re-fold the corners, this time folding all layers



The final piece:



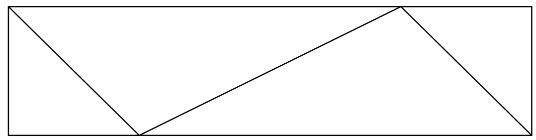
Why?



BC = 2
AC = CD = 1
BAC = atan(BC/AC) = 63.44
CAD = 45
BAD = 63.44+45 = 108.44

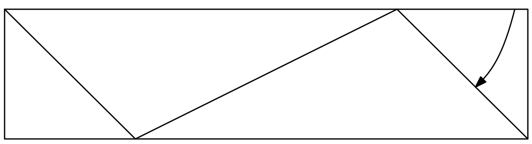
Hexagon Module (120 Degrees)

Get to step 8 of the Pentagon module:

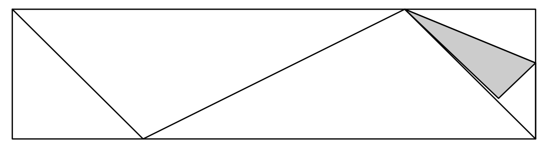


1

Fold the top to the diagonal line (Fold the top layer only):

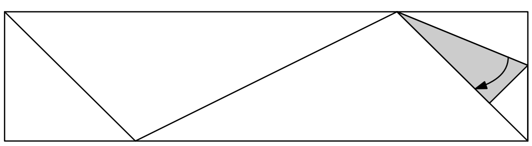


2

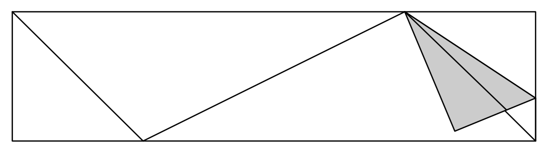


3

Fold this new crease to the diagonal, opening as you fold:

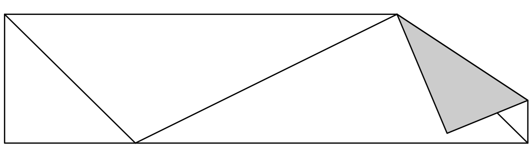


4



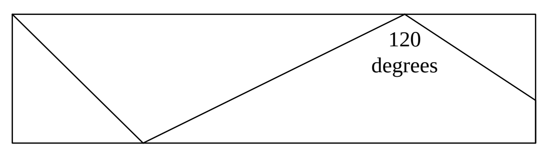
5

Fold all layers along this new crease:



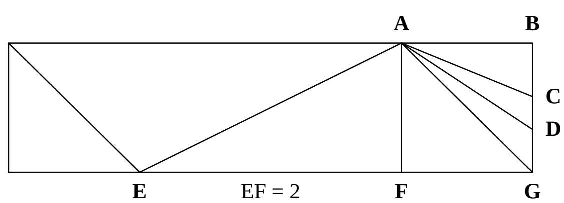
6

Open up. Final fold:



7

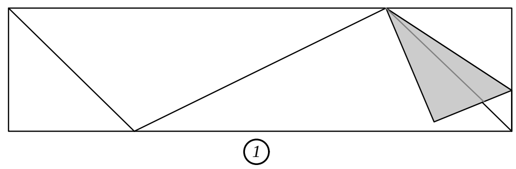
Why?



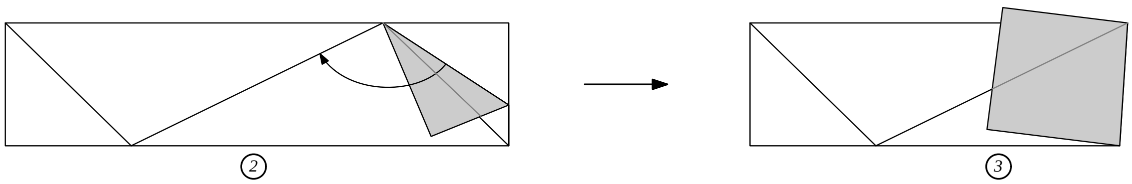
AF = AB = BG = 1
EAF = atan(EF/AF) = 63.44
GAF = 45
GAC = 45/2 = 22.5
DAC = 22.5/2 = 11.25
EAD = 63.44 + 45 + 11.25 = 119.69 (almost 120)

Triangle Module (60 Degrees)

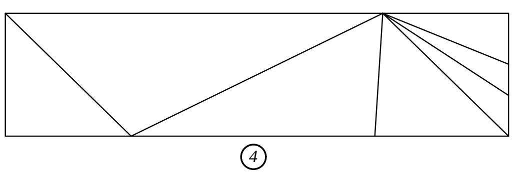
Get to step 5 of the Hexagon module:



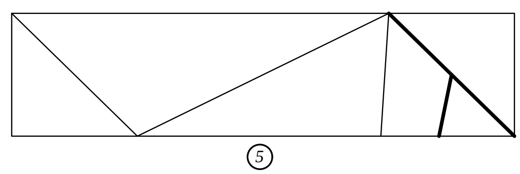
Fold the entire module so that this newly created crease matches up with the large crease:



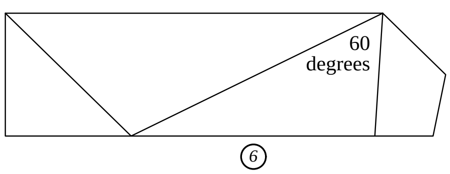
Unfold back to the rectangle:



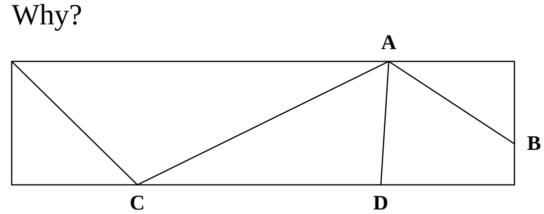
Cut along the thick lines (one of them is not along any crease lines)



The final piece:



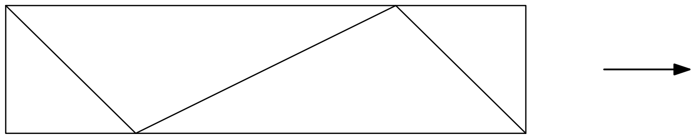
Why?



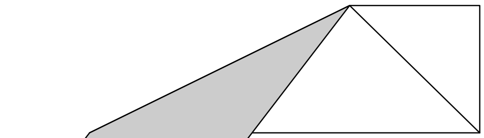
CAB = 119.69 degrees (from the Hexagon module)
CAD = CAB/2 = 59.85 (almost 60)

Square Module (90 Degrees)

Get to step 8 of the Pentagon module, and fold along the long crease

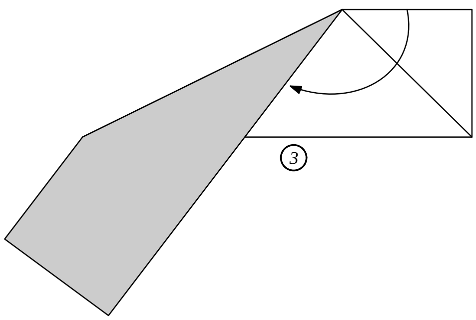


1

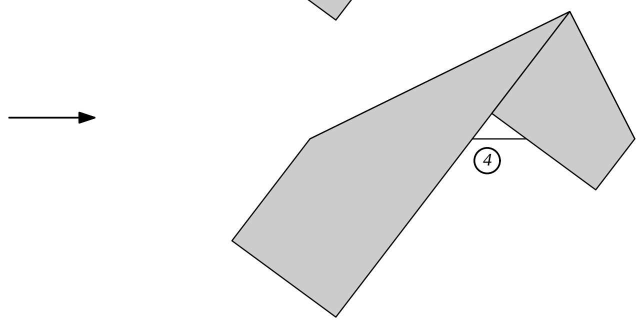


2

Fold the entire piece as indicated:

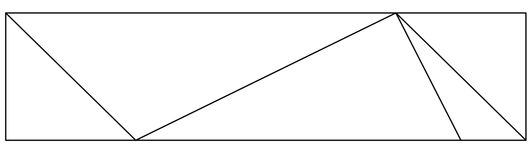


3



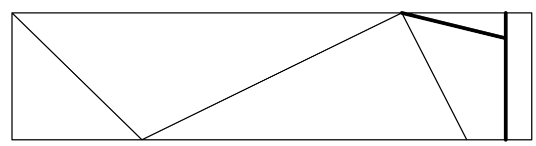
4

Unfold back to the rectangle:



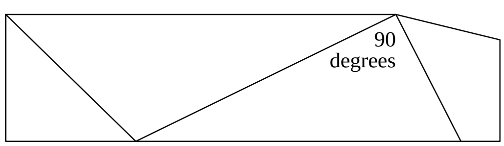
5

Cut along the thick lines (these are not along any crease lines)



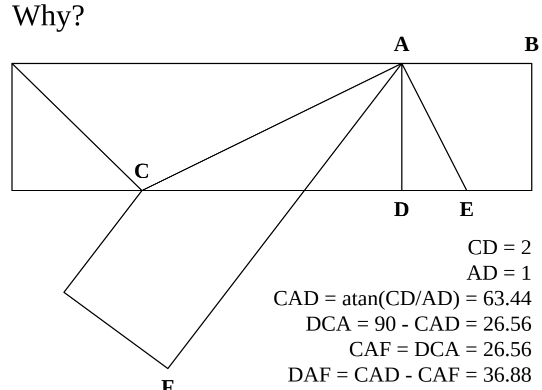
6

The final piece:



7

Why?



CD = 2
AD = 1
CAD = atan(CD/AD) = 63.44
DCA = 90 - CAD = 26.56
CAF = DCA = 26.56
DAF = CAD - CAF = 36.88
FAB = 90 + DAF = 126.88
FAE = FAB/2 = 63.44
CAE = FAE + CAF = 90